

# Estimating Natural Frequencies and Mode Shapes from Forced Response Calculations

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A method is proposed for condensing a structural eigenvalue problem by using more than one forced response vector in a condensation. The method approximates an eigenvector by a linear combination of forced response vectors, resulting in a significantly smaller eigenvalue problem that retains the eigenpairs that were most excited by the force vectors. The condensation is extremely economical in cases where the forced response vectors have been computed for other purposes. In cases where some eigenpairs are approximately known, the method produces accurate eigenvalue estimates by choosing the force vectors to be parallel to the approximate eigenvectors. This idea forms the basis of an iterative procedure that finds all of the eigenvalues in a specified frequency range. Theoretical error bounds on the estimated eigenvalues are also presented. Numerical examples illustrate unique features of the condensation in its noniterative and iterative forms.

## Nomenclature

$A$	= system matrix, $K - \lambda M$
$C_n$	= participation factor of the $n$ th mode
$d$	= vector of amplitudes in eigenvector approximation
$K$	= stiffness matrix
$M$	= mass matrix
$P$	= matrix of primary amplitudes
$R$	= Rayleigh's quotient
$r$	= residual error in eigenvalue problem
$S$	= matrix of secondary amplitudes
$X$	= matrix holding forced response vectors
$x$	= forced response vector
$\epsilon$	= scaling factor for error in eigenvector estimate
$\kappa$	= condensed stiffness matrix
$\Lambda_P$	= diagonal matrix of primary eigenvalues
$\Lambda_S$	= diagonal matrix of secondary eigenvalues
$\lambda$	= frequency parameter, $\omega^2$
$\lambda_n$	= $n$ th eigenvalue
$\tilde{\lambda}_n$	= approximation of the $n$ th eigenvalue
$\mu$	= condensed mass matrix
$\Phi$	= matrix holding eigenvectors
$\tilde{\Phi}$	= matrix holding approximate eigenvectors
$\phi_n$	= $n$ th eigenvector
$\tilde{\phi}_n$	= approximation of the $n$ th eigenvector
$\psi^{(0)}$	= shape of error in eigenvector estimate
$\omega$	= radian frequency

## Subscripts

max	= maximum value
min	= minimum value
m	= forced response index
n	= eigenpair index

## Superscript

(p)	= iterate
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## Introduction

MANY analysts are concerned with the free and forced vibrations predicted by structural models over specified frequency ranges. To estimate a structure's frequency response, for example, an engineer may compute the forced response of a structure at a number of frequencies in a given frequency range. Given this response, the engineer is then faced with the question, "Which of the myriad of vibrational modes present in the structure are primarily responsible for the observed vibration pattern?" This paper presents an efficient and straightforward postprocessing method that answers this question directly. The method can also be used as the kernel of an iterative scheme to find all vibrational modes in a given frequency band. We present both applications here.

The present method belongs to a class of methods in which an eigenvector is approximated by a linear combination of basis vectors. Use of this approximation in Hamilton's principle is often referred to as the Rayleigh-Ritz method, and the vectors are known as Ritz vectors or assumed modes. When inserted into the full eigenvalue problem, the approximation produces a much smaller eigenvalue problem that retains approximations of the desired eigenpairs. The distinguishing characteristic among methods is the choice of basis vectors. Thorough and interesting discussions of the Rayleigh-Ritz method in eigenvalue problems are found in the texts by Hughes<sup>1</sup> and Bathe.<sup>2</sup>

One of the earliest expressions of the idea is found in Crandall's monograph,<sup>3</sup> which was published in 1956. In describing a truncated Lanczos method, he proposed a procedure for estimating eigenpairs by expanding an eigenvector in a series of orthogonal vectors. Requiring stationarity of Rayleigh's quotient with respect to the coefficients in the expansion created a reduced eigenvalue problem that Crandall referred to as an "... eigenvalue problem within an eigenvalue problem." In the same year, Turner et al.<sup>4</sup> presented a method for condensing structural matrices by expanding displacements in polynomial basis functions. This expansion in kinetic and strain energy expressions yielded reduced matrices of the same form as those in Crandall's reduced eigenvalue problem.<sup>3</sup>

Since that time, a variety of methods have been developed based on alternate choices of basis vectors. Bhat<sup>5</sup> chose basis vectors that satisfied the geometric and natural boundary conditions on the structure. Matta<sup>6</sup> proposed a rationale for selecting other basis vectors based on the diagonal terms of the mass and stiffness matrices. Other researchers prescribed basis vectors that identified and removed unimportant coordinates. At the time, such methods were not recognized as Rayleigh-Ritz methods. For example, the mass condensation method developed by Irons<sup>7,8</sup> and Guyan<sup>9</sup> in

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the 1960s was interpreted as a Rayleigh–Ritz method by Fried<sup>10</sup> in 1972 and the basis vectors in the method were identified by Matta<sup>6</sup> in 1987. These recognitions were useful because they allowed the use of convergence characteristics and error estimates derived for the Rayleigh–Ritz method (see Refs. 11 and 12).

The primary distinction between the present method and previous methods is the use of forced response vectors as basis vectors. In this paper, the forced response vectors are used to extract eigenpair estimates based on the simultaneous consideration of the proximity of an eigenvalue to the square of the forcing frequency and the projection of the eigenvector on the forcing vector. Therefore, the condensation retains the subset of eigenpairs with the greatest participation in the forced response. As a consequence, the method may be used to extract eigenvalues that lie outside a frequency range of interest if their participation in the forced response is large enough.

An iterative implementation of the present method may be viewed as an extension of Rayleigh quotient iteration (see Ref. 13) (RQI), and we shall, therefore, refer to it as multifrequency RQI (MRQI). In RQI, an eigenvector estimate is used in Rayleigh's quotient to achieve a better estimate of the eigenvalue. Next, the system is forced at the eigenvalue estimate using the eigenvector estimate as a force vector. The normalized response vector is used as a new eigenvector estimate, and the process is repeated. The MRQI method proposed here follows this general procedure; however, many eigenvectors and eigenvalues are simultaneously considered in the iteration. Eigenpair estimates produced by the present method may also be refined by iterative methods that avoid the matrix inversions incurred in RQI. Discussions of such methods may be found in various texts.<sup>14–16</sup>

In the following section, the postprocessing of forced response vectors in a Rayleigh–Ritz method is described, and its properties are examined. Next, the motivation to use this method as the basis of iterative solution of the eigenvalue problem is presented by reviewing the convergence characteristics of RQI. So motivated, we give an implementation of the MRQI, and error estimates for the eigenvalues, which may be used at any stage of the iteration, are given. This implementation is designed to find all of the eigenvalues in a specified frequency range. Finally, numerical examples that illustrate the properties of the method in its noniterative and iterative forms are presented, and conclusions are drawn regarding possible applications and extensions of the method.

### Forced Response Vectors in the Rayleigh–Ritz Method

We consider the class of structural vibration problems that is described by the matrix equation

$$[\mathbf{K} - \omega^2 \mathbf{M}] \mathbf{x} = \mathbf{f} \quad (1)$$

In Eq. (1),  $\mathbf{K}$  is the symmetric positive semidefinite stiffness matrix,  $\mathbf{M}$  is the symmetric positive definite mass matrix,  $\mathbf{x}$  is the vibrational response vector at frequency  $\omega$ , and  $\mathbf{f}$  is the forcing vector, which may in general depend on  $\omega$ . The sizes of the matrices in Eq. (1) are  $N \times N$ .

The modes of vibration of the structure under consideration are defined in terms of the following generalized eigenvalue problem:

$$[\mathbf{K} - \lambda_n \mathbf{M}] \phi_n = \mathbf{0} \quad (2)$$

Here,  $\lambda_n = \omega_n^2$  is the square of the  $n$ th natural frequency, and  $\phi_n$  is the  $n$ th eigenvector or modeshape. We assume that there are no repeated eigenpairs. In all that follows, we shall assume that the eigenvectors are normalized according to

$$\phi_n^T \mathbf{M} \phi_n = \delta_{mn} \quad (3)$$

where  $\delta_{mn}$  is the Kronecker delta. For convenience of notation, we define a system matrix according to

$$\mathbf{A}(\lambda) = \mathbf{K} - \lambda \mathbf{M} \quad (4)$$

We suppose that we have solved Eq. (1) at  $M$  different frequencies in a frequency band, to predict the frequency response of the structure in that band. We seek then to approximate the  $M$  out of

the total  $N$  vibrational modes that most contribute to the response in the chosen frequency band. To that end, we recast Eq. (1) in the form

$$\mathbf{A}(\lambda) \mathbf{x}(\lambda) = \mathbf{f}(\lambda) \quad (5)$$

where  $\mathbf{x}$  is the response vector. The frequency parameter  $\lambda$  is related to the forcing frequency by  $\lambda = \omega^2$ . To obtain eigenvalue and eigenvector estimates, we will use a Rayleigh–Ritz method. The basis vectors to be used in the Rayleigh–Ritz reduction are chosen to be the response vectors found by solving Eq. (5) at discrete frequencies:

$$\mathbf{x}_m = \mathbf{A}^{-1}(\tilde{\lambda}_m) \mathbf{f}_m \quad (6)$$

where  $\mathbf{f}_m = \mathbf{f}(\tilde{\lambda}_m)$ .

We now let the eigenvector be approximated by a linear combination of the forced response vectors:

$$\phi \approx \sum_{m=1}^M \mathbf{x}_m d_m = \mathbf{X} \mathbf{d} \quad (7)$$

where  $\mathbf{X}$  is a transformation matrix whose columns are the  $\mathbf{x}_m$  vectors and  $\mathbf{d}$  is a vector holding the  $d_m$  coefficients.

Substituting this approximation into the eigenvalue problem gives

$$\mathbf{A}(\lambda) \mathbf{X} \mathbf{d} = \mathbf{r} \quad (8)$$

where the residual vector  $\mathbf{r}$  results from the approximation of  $\phi$  in Eq. (7). The requirement that the residual vector be orthogonal to the basis vectors is satisfied by premultiplying by  $\mathbf{X}^T$  and setting the right-hand side to zero. This gives the transformed eigenvalue problem

$$(\kappa - \lambda \mu) \mathbf{d} = \mathbf{0} \quad (9)$$

where the transformed mass and stiffness matrices are

$$\mu = \mathbf{X}^T \mathbf{M} \mathbf{X}, \quad \kappa = \mathbf{X}^T \mathbf{K} \mathbf{X} \quad (10)$$

Once the eigenvalue problem in Eq. (9) is solved for the eigenpairs  $\lambda_n$  and  $\mathbf{d}_n$ , the full eigenvectors are approximated by

$$\phi_n \approx \mathbf{X} \mathbf{d}_n \quad (11)$$

whereas the eigenvalues of the full problem are approximated by the eigenvalues of the reduced problem.

To support the claim that these are the eigenpairs most excited by the force, we consider the case  $M = 1$ . The response vector is given by

$$\mathbf{x}_1 = \sum_{n=1}^N C_n \phi_n \quad (12)$$

where the participation factor  $C_n$  quantifies the importance of the  $n$ th eigenpair to the forced response and is given by

$$C_n = \phi_n^T \mathbf{f}_1 / (\lambda_n - \tilde{\lambda}_1) \quad (13)$$

The numerator of  $C_n$  measures the projection of the force vector onto each eigenvector and the denominator measures the proximity of  $\tilde{\lambda}_1$  to each eigenvalue.

When Eq. (12) is used to transform the eigenvalue problem according to Eq. (9), an eigenvalue  $\lambda_p$  is estimated as

$$\lambda_p \approx \frac{\sum_{n=1}^N C_n^2 \lambda_n}{\sum_{n=1}^N C_n^2} \quad (14)$$

This equation was presented by Crandall,<sup>3</sup> who observed that the eigenvalue estimate  $\lambda_j$  is a weighted average of all of the eigenvalues. From Eq. (13), we observe that the weighting factors for this method are determined by the force distribution  $\mathbf{f}_1$  and the frequency parameter  $\tilde{\lambda}_1$ . For Eq. (14) to yield an accurate estimate of  $\lambda_j$ , the forcing vector should be more parallel to  $\phi_j$  and  $\tilde{\lambda}_1$  should be closer to  $\lambda_j$  than for any other eigenvalue.

This finding is generalized by writing the forced response vectors in Eq. (6) as linear combinations of eigenvectors:

$$\mathbf{x}_m = \sum_{n=1}^N C_{nm} \phi_n \quad (15)$$

where the amplitudes are found by generalizing Eq. (13):

$$C_{nm} = \phi_n^T \mathbf{f}_m / (\lambda_n - \tilde{\lambda}_m) \quad (16)$$

Writing Eq. (15) into a matrix equation gives

$$\mathbf{X} = \Phi \mathbf{C} \quad (17)$$

Substituting this result into the expansion in Eq. (7) yields

$$\phi_n \approx \Phi \mathbf{C} \mathbf{d} \quad (18)$$

If this approximation is used to condense the eigenvalue problem, one arrives at Eq. (9), but the condensed mass and stiffness matrices are now expressed in terms of the modal amplitude matrix

$$\mu = \mathbf{C}^T \mathbf{C} \quad (19)$$

$$\kappa = \mathbf{C}^T \Lambda \mathbf{C} \quad (20)$$

where  $\Lambda$  is a diagonal matrix that contains the eigenvalues.

This form of the condensation can be used to illustrate its ability to retain a subset of eigenpairs that are most excited over the frequency band of interest by the chosen force vectors. To see this, let us assume that a subset of eigenpairs, henceforth referred to as the primary eigenpairs, contribute much more to the response vectors than do a complementary set of secondary eigenpairs. Then the amplitude matrix may be partitioned according to

$$\mathbf{C} = \begin{bmatrix} \mathbf{P} \\ \mathbf{S} \end{bmatrix} \quad (21)$$

where  $\mathbf{P}$  and  $\mathbf{S}$  hold the primary and secondary amplitudes. When this partitioned form is used, the condensed matrices have the form

$$\kappa = \mathbf{P}^T \Lambda_P \mathbf{P} + \mathbf{S}^T \Lambda_S \mathbf{S} \quad (22)$$

$$\mu = \mathbf{P}^T \mathbf{P} + \mathbf{S}^T \mathbf{S} \quad (23)$$

where  $\Lambda_P$  and  $\Lambda_S$  are diagonal matrices that contain the primary and secondary eigenvalues.

If the amplitudes of the primary and secondary eigenpairs satisfy the inequalities

$$\mathbf{P}^T \Lambda_P \mathbf{P} \gg \mathbf{S}^T \Lambda_S \mathbf{S} \quad (24)$$

$$\mathbf{P}^T \mathbf{P} \gg \mathbf{S}^T \mathbf{S} \quad (25)$$

then the condensed eigenvalue problem is approximated by

$$\mathbf{P}^T (\Lambda_P - \lambda \mathbf{I}) \mathbf{P} \mathbf{d} \approx \mathbf{0} \quad (26)$$

By inspection, the eigenpairs of this eigenvalue problem are approximately the primary ones from the original eigenvalue problem. If the secondary modal amplitudes are identically zero, the condensation identically retains the primary eigenpairs. In practice, estimates of the primary eigenpairs improve as the their associated amplitudes become much greater than those of the secondary eigenpairs.

The method just presented extracts  $M$  eigenpairs estimates, but not all of the estimates are necessarily good estimates in practice. To evaluate the accuracy of the eigenpairs estimates, we have developed an error bound (and estimate), which is presented in a later section. If the errors are found to be too great, one might wish to improve the eigenpair estimates. For that purpose, the method just presented can be used iteratively as shown in the next section.

## Iterative Improvement: Motivation

Most algorithms to find eigenvalues of a matrix  $\mathbf{A}$  require its inverse. To see why this is so, we consider the particular form of  $\mathbf{A}(\lambda)$  given in Eq. (4) and write the inverse of  $\mathbf{A}$  in terms of its eigenvectors and eigenvalues defined in Eq. (2):

$$\mathbf{A}^{-1}(\lambda) = \sum_{n=1}^N \frac{1}{\lambda_n - \lambda} \phi_n \phi_n^T \quad (27)$$

Equation (27) shows that the inverse matrix  $\mathbf{A}^{-1}$  is dominated by the eigenvector product  $\phi_n \phi_n^T$  if  $\lambda_n$  is the eigenvalue closest to  $\lambda$ . If this is so, operating on a vector by the matrix  $\mathbf{A}^{-1}$  tends to project that vector toward  $\phi_n$ . This is the principle behind the shifted inverse power method to find interior eigenvalues and eigenvectors. It is also the primary reason the proposed method works so well in just a single application.

## RQI

RQI and inversion can be used to accelerate convergence toward a particular eigenpair. Suppose we begin with a vector  $\tilde{\phi}_n^{(0)} = \phi_n + \epsilon \psi^{(0)}$ , where  $\epsilon$  is a small parameter. The vector  $\tilde{\phi}_n$  represents a known approximation of the unknown exact eigenvector  $\phi_n$ . The superscript denotes the iterate, whereas the subscript identifies the eigenvector of interest. From  $\tilde{\phi}_n^{(0)}$ , we can obtain the following approximation to  $\lambda_n$  using Rayleigh's quotient:

$$\tilde{\lambda}_n^{(0)} = R(\tilde{\phi}_n^{(0)}) \quad \text{where} \quad R(\phi) = \phi^T \mathbf{K} \phi / \phi^T \mathbf{M} \phi \quad (28)$$

Applying Rayleigh's quotient yields a second-order approximation for the eigenvalue. That is, for some constant  $C_n$  independent of  $\epsilon$ ,

$$\tilde{\lambda}_n^{(0)} = \lambda_n + \epsilon^2 C_n + \mathcal{O}(\epsilon^4) \quad (29)$$

Thus, a first-order estimate of the eigenvector gives a second-order estimate of the eigenvalue by Rayleigh's quotient.

The iteration is defined as

$$\lambda_n^{(p)} = R(\tilde{\phi}_n^{(p)}) \quad (30)$$

$$\tilde{\phi}_n^{(p+1)} = \mathbf{A}^{-1}(\lambda_n^{(p)}) \mathbf{M} \tilde{\phi}_n^{(p)} \quad (31)$$

To see the result of iteration, we use Eqs. (3), (27), and (29) in Eq. (31):

$$\tilde{\phi}_n^{(p+1)} = \frac{1}{\epsilon^2 C_n} \phi_n + \epsilon \sum_{m \neq n} \frac{1}{\lambda_m - \tilde{\lambda}_n^{(0)}} \phi_m \phi_m^T \mathbf{M} \psi^{(0)} \quad (32)$$

$$\tilde{\phi}_n^{(p+1)} = \frac{1}{\epsilon^2 C_n} \{ \phi_n + \epsilon^3 \psi^{(1)} + \mathcal{O}(\epsilon^4) \} \quad (33)$$

Equation (33) shows that one iteration starting with an estimate within order  $\epsilon$  of  $\phi_n$  yields an estimate within order  $\epsilon^3$  of  $\phi_n$ . That is, once the algorithm gets close to a solution of the eigenvalue problem, it can converge to it very quickly. Before it gets into a basin of attraction of one of the eigenvalues, however, it may skip around significantly.<sup>13</sup> This can result in convergence to an eigenpair that is far from that which the analyst hoped to find.

## Convergence to Unexpected Eigenpairs

To understand the convergence to unexpected eigenvalues, let us consider the action of  $\mathbf{A}^{-1}$  on a vector that is a combination of two eigenvectors. Let us begin with an eigenvector estimate of

$$\tilde{\phi}^{(0)} = \alpha \mathbf{M} \phi_m + \beta \mathbf{M} \phi_n \quad (34)$$

and an eigenvalue estimate of

$$\tilde{\lambda}^{(0)} = \lambda_n (1 - \epsilon) \quad (35)$$

Then, by Eq. (27) we obtain

$$\tilde{\phi}^{(1)} = \mathbf{A}^{-1}(\tilde{\lambda}^{(0)}) \tilde{\phi}^{(0)} = \{ \alpha / [\lambda_m - \lambda_n (1 - \epsilon)] \} \phi_m + (\beta / \lambda_n \epsilon) \phi_n \quad (36)$$

Equation (36) shows that as  $\epsilon \rightarrow 0$ ,  $\tilde{\phi}^{(1)}$  is dominated by the contribution from  $\phi_n$ . However, for finite values of  $\epsilon$ , vector  $\tilde{\phi}^{(1)}$  need not be dominated by  $\phi_n$ . For example, if  $\beta = \mathcal{O}(\epsilon)$  (i.e., our initial eigenvector estimate has small projection in the  $\phi_n$  direction), then  $\phi^{(1)}$  can be dominated by  $\phi_m$ . In fact, the condition for  $\tilde{\phi}$  to be dominated by the  $\phi_n$  eigenvector is

$$\{\alpha/[\lambda_m - \lambda_n(1 - \epsilon)]\} \ll \beta/\lambda_n \epsilon \quad (37)$$

Even if  $\epsilon$  is very small, Eq. (37) only will be satisfied provided that  $\beta$  is large enough and  $\alpha$  is not too large. If Eq. (37) is not satisfied, then RQI can converge to  $(\lambda_m, \phi_m)$  rather than  $(\lambda_n, \phi_n)$ , despite having initialized the iteration with a guess of  $\lambda$  close to  $\lambda_n$ .

### MRQI

Each step in RQI can be interpreted as a special case of the forced condensation method with just a single force vector. The natural extension, therefore, is to use several forced response vectors iteratively to improve eigenpair estimates. Here we describe one straightforward implementation of MRQI that is designed to find all of the eigenvalues in a selected frequency range.

Extending the forced response condensation from a single step reduction to an iterative method introduces two main technical issues. First, suppose the number of eigenvalues in the specified range is  $P + Q$ . Because the method described finds  $M$  eigenvectors, we need to ensure that  $M \geq P + Q$ . The way to do this a priori is described next. The second issue is that the iteration must be terminated at some point. The decision to terminate must be based on an error estimate of the eigenpair accuracy. An error estimate to accomplish that goal is presented in the following section.

### Iteration

At the start of each iteration, we have a vector of  $Q$  approximate eigenvalues,  $\lambda_q$ ,  $q = 1, \dots, Q$ , arranged in ascending order. Compatibly sorted is a matrix of  $Q$  approximate eigenvectors. We have also a matrix of  $P$  converged eigenvectors:

$$\tilde{\Phi} = [\tilde{\phi}_1, \tilde{\phi}_2, \dots, \tilde{\phi}_Q] \quad (38)$$

$$\Phi = [\phi_1, \phi_2, \dots, \phi_P] \quad (39)$$

The  $\Phi$  are assumed to be orthonormal with respect to  $\mathbf{M}$ . As the iteration progresses, the number of unknown eigenvectors decreases while the number of converged eigenvectors increases.

Following the RQI in Eq. (31), we now define the new approximate eigenvectors as

$$\tilde{\phi}_q \leftarrow A^{-1}(\tilde{\lambda}_q) \mathbf{M} \tilde{\phi}_q \quad q = 1, \dots, Q \quad (40)$$

$$\tilde{\Phi} \leftarrow [\mathbf{I} - \Phi \Phi^T \mathbf{M}] \tilde{\Phi} \quad (41)$$

The step in Eq. (41) prevents convergence to previously found eigenpairs by removing any components of  $\tilde{\phi}_q$  in the directions of the converged subspace.<sup>2</sup> The reduced mass and stiffness matrices are computed according to Eq. (10), where the columns of the transformation matrix  $\mathbf{T}$  are taken as the current eigenvector estimates  $\tilde{\phi}_q$ . The reduced eigenvalue problem in Eq. (9) is solved for new estimates of the eigenvectors and eigenvalues.

Finally, we update and sort our results. The updates to the eigenvalues are the eigenvalues determined from solving Eq. (9). The updates to the eigenvectors are given by our transformation:

$$\tilde{\phi}_q \leftarrow \mathbf{X} \mathbf{d}_q \quad q = 1, \dots, Q \quad (42)$$

### Sorting

Before starting the next iteration, several sorting steps are required. We first note that, if we make the  $\alpha_q$  orthonormal with respect to  $\mu$ , then the resulting  $\mathbf{x}_q$  will be orthonormal with respect to  $\mathbf{M}$ . This is a useful observation because working with the small vectors is significantly more efficient than working with the larger vectors.

Next, the eigenpairs are sorted into converged and approximate. This sorting is based on eigenvalue error bounds presented in the

following section. If an eigenpair is determined to be converged, then its eigenvector is removed from the matrix  $\tilde{\Phi}$  and added to the matrix  $\Phi$ . The values of  $P$  and  $Q$  are appropriately incremented and decremented. For all future iterations, the trial vectors shall be made orthogonal to the newly converged eigenvector. Of course, the converged eigenvalue is removed from the vector of eigenvalue estimates and saved in an output vector.

The next check is to determine whether the eigenvalue is inside the desired interval. If so, we are finished with the iteration. If not, however, then we must increment  $Q$  and add a new eigenpair estimate. The choice of this new eigenpair estimate may be guided by Sylvester's inertia theorem, which is described next.

### Initialization

The MRQI method may be initialized with eigenvalue estimates that favor the convergence to eigenvalues within a frequency range by using Sylvester's inertia theorem. In fact, Sylvester's inertia theorem can be used as a basis to find eigenvalues by the method of bisection (see Ref. 16, Chap. 3), but RQI will usually converge much faster. The initialization procedure described here places a frequency parameter at each end of the range and distributes the remaining frequency parameters as near as possible to eigenvalues in the band. As discussed earlier, this does not guarantee convergence to all of the eigenvalues in the band because the converged values depend on the force vector as well as the proximity of a frequency parameter to an eigenvalue.

Let the range be defined by minimum and maximum values of the frequency parameter,  $\lambda_{\min}$  and  $\lambda_{\max}$ , and let the analyst choose a force vector that, in general, depends on  $\lambda$ . Then the first and second frequency parameters are positioned at the ends of the range, so that  $\tilde{\lambda}_1 = \lambda_{\min}$  and  $\tilde{\lambda}_2 = \lambda_{\max}$ . Equation (5) is solved for the approximate eigenvectors corresponding to the choices

$$\tilde{\phi}_1 = A^{-1}(\tilde{\lambda}_1) \mathbf{f}_1 \quad (43)$$

$$\tilde{\phi}_2 = A^{-1}(\tilde{\lambda}_2) \mathbf{f}_2 \quad (44)$$

In addition, let  $S(\tilde{\lambda}_q)$  denote the number of roots of  $A(\lambda) = 0$  that are less than  $\tilde{\lambda}_q$ . If we solve Eqs. (43) and (44) by  $LDL^T$  factorization (see Ref. 14, Sec. 4.2), then we can easily evaluate  $S(\tilde{\lambda}_q)$  by counting the number of negative entries on the diagonal of  $D$  at each factorization. Thus, each time we evaluate  $A^{-1}(\tilde{\lambda}_q)$ , we can simultaneously evaluate  $S(\tilde{\lambda}_q)$  for practically no cost. By solving Eqs. (43) and (44), we know that the total number of eigenvalues in the range to be found is  $Q = S(\tilde{\lambda}_2) - S(\tilde{\lambda}_1)$  and, because this is the first iteration,  $P = 0$ .

This eigenvalue counting is used to position recursively the remaining  $Q - 2$  frequency parameters in the range so that they lie as close as possible to the eigenvalues in the range. The third parameter is positioned halfway between the first and second,  $\tilde{\lambda}_3 = (\tilde{\lambda}_1 + \tilde{\lambda}_2)/2$ , creating two half-ranges. The eigenvalues in the two half-ranges are counted, and the fourth parameter is chosen to bisect the half-range with the most eigenvalues. This process continues, with each new frequency parameter chosen to bisect the partial range with the most eigenvalues. This initialization scheme has two important properties. In the special case where the eigenvalues are evenly spaced in the range, a frequency parameter will be inserted between each pair of adjacent eigenvalues. In the special case where the eigenvalues are clustered in a subrange, more frequency parameters will be selected near or inside the subrange.

If all of the eigenvalues are tightly clustered, the algorithm will probably not place a frequency parameter between any of the eigenvalues. If this happens, one must decide whether to continue with the bisection or to begin the iteration. One advantage of beginning the iteration is that the condensation will likely locate the region of the clustering with one iteration, whereas simple bisection might require evaluations at many frequency parameters to locate the region.

### Theoretical Error Estimates

In this section, we derive an error bound for a given approximation to a natural frequency and mode shape. We believe that this bound is

new for the generalized eigenvalue problem considered in this paper. To obtain the bound, we first need to introduce some nomenclature. In the generalized eigenvalue problem, it is natural to measure the norms of vectors using a mass-weighted norm. Thus, we introduce the norm of  $\mathbf{x}$  as

$$\|\mathbf{x}\|_M = \sqrt{\mathbf{x}^T \mathbf{M} \mathbf{x}} \quad (45)$$

The following result is critical to the derivation:

$$\|A^{-1} \mathbf{M} \mathbf{x}\|_M \leq 1 / \left( \min_n |\lambda_n - \lambda| \right) \|\mathbf{x}\|_M \quad (46)$$

To prove Eq. (46), we let

$$\mathbf{x} = \sum_{n=1}^N a_n \phi_n$$

and use Eq. (27) to write

$$A^{-1} \mathbf{M} \mathbf{x} = \sum_{n=1}^N \frac{a_n}{\lambda_n - \lambda} \phi_n \quad (47)$$

Taking mass-weighted norms of both sides of Eq. (47) leads to

$$\begin{aligned} \|A^{-1} \mathbf{M} \mathbf{x}\|_M^2 &= \sum_{n=1}^N \frac{a_n}{\lambda_n - \lambda} \phi_n^T \mathbf{M} \sum_{m=1}^N \frac{a_m}{\lambda_m - \lambda} \phi_m \\ &= \sum_{n=1}^N \frac{a_n^2}{(\lambda_n - \lambda)^2} \leq \frac{1}{\min_n (\lambda_n - \lambda)^2} \sum_{n=1}^N a_n^2 \\ &\leq \frac{1}{\min_n (\lambda_n - \lambda)^2} \|\mathbf{x}\|_M^2 \end{aligned} \quad (48)$$

We see that Eq. (48) follows as the square root of Eq. (48).

Now, suppose  $\tilde{\lambda}$  and  $\tilde{\phi}$  are given as approximations of an exact eigenpair and we wish to bound the error in the eigenvalue estimate. First, note the identity

$$\tilde{\phi} = (\mathbf{K} - \tilde{\lambda} \mathbf{M})^{-1} \mathbf{M} \mathbf{M}^{-1} (\mathbf{K} - \tilde{\lambda} \mathbf{M}) \tilde{\phi} \quad (49)$$

Taking mass-weighted norms of both sides of Eq. (48) and using Eq. (49) gives

$$\begin{aligned} \|\tilde{\phi}\|_M &= \|(\mathbf{K} - \tilde{\lambda} \mathbf{M})^{-1} \mathbf{M} \mathbf{M}^{-1} (\mathbf{K} - \tilde{\lambda} \mathbf{M}) \tilde{\phi}\|_M \\ &\leq 1 / \left( \min_n |\lambda_n - \tilde{\lambda}| \right) \|\mathbf{M}^{-1} (\mathbf{K} - \tilde{\lambda} \mathbf{M}) \tilde{\phi}\|_M \end{aligned} \quad (50)$$

Rearranging Eq. (50) gives an error bound on the eigenvalue estimate

$$|\lambda_n - \tilde{\lambda}| \leq \frac{\|\mathbf{M}^{-1} (\mathbf{K} - \tilde{\lambda} \mathbf{M}) \tilde{\phi}\|_M}{\|\tilde{\phi}\|_M} \quad (51)$$

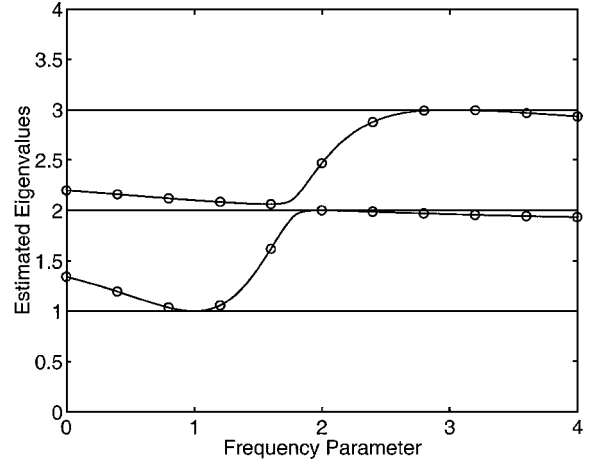
Given  $\tilde{\lambda}$  and  $\tilde{\phi}$ , the right-hand side of Eq. (51) is easily evaluated. The result quantifies how well the estimated eigenpair satisfies the eigenvalue problem.

### Numerical Examples

In this section we examine a set of examples to demonstrate features of the forced response condensation. The examples are arranged in order of increasing complexity, with the first examples illustrating specific features of the condensation and the final examples indicating general trends. The first example is a three-degree-of-freedom system that has eigenvalues  $\lambda_n = n$ , where  $n = 1, 2, 3$ . The orthonormal eigenvectors are  $\phi_n = \mathbf{e}_n$ , where  $\mathbf{e}_n$  is a vector whose  $n$ th element is unity and all other elements are zero. The corresponding mass and stiffness matrices are found by inverting Eqs. (2) and (3):

$$\mathbf{K} = (\Phi^T)^{-1} \Phi^{-1} \quad (52)$$

$$\mathbf{M} = (\Phi^T)^{-1} \text{diag}(\lambda) \Phi^{-1} \quad (53)$$



**Fig. 1** Eigenvalues estimated by a condensation with one frequency parameter at  $\lambda = 2.25$  and the other parameter varied along the abscissa; solid horizontal lines represent exact eigenvalues.

Now suppose that two eigenpairs are to be estimated by computing the forced response at two frequency parameters. This example will show how the estimated eigenpairs depend on the chosen frequency parameter and force vector. One of the frequency parameters is fixed at  $\lambda = 2.25$ , and the other is varied. For each frequency parameter, the force vector is chosen to equally excite all three eigenpairs:

$$\mathbf{f} = \sum_{n=1}^N \mathbf{M} \phi_n \quad (54)$$

The eigenvalue problem is reduced according to Eq. (10) and two eigenpairs are estimated.

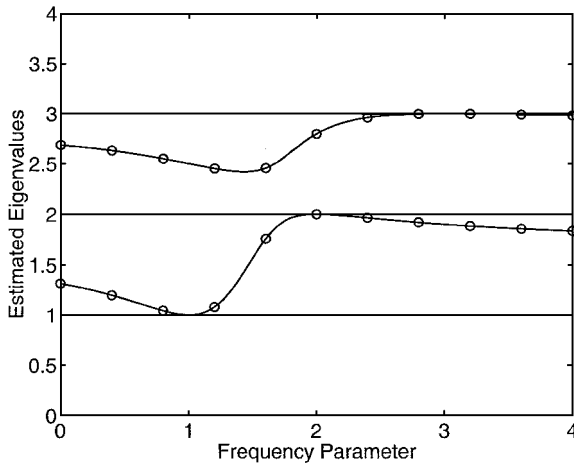
The two estimated eigenvalues are plotted in Fig. 1 vs the varied frequency parameter. In Fig. 1, the force vector equally excites all eigenpairs. One of the estimated eigenvalues is within 10% of  $\lambda_2$  for any choice of the varied frequency and is very close to  $\lambda_2$  when the varied frequency parameter is near  $\lambda_2$ . This follows from Eq. (27) because the forced response vector is dominated by the eigenpair when the frequency parameter is near the eigenvalue. Note, however, that the other two eigenpairs are only approximated when the frequency parameter is near their eigenvalues. The situation is made worse when the fixed frequency parameter is halfway between two eigenvalues,  $\lambda = 2.5$ . Using this fixed parameter and varying the other one as before results in Fig. 2. Again the force vector equally excites all eigenpairs. We notice that  $\lambda_2$  is now only approximated when the frequency parameter is near it. As before,  $\lambda_1$  and  $\lambda_3$  are also only approximated when the frequency parameter is near them.

This problem may be solved by choosing a force vector that is a linear combination of two desired eigenvectors. In actual applications of the method, this choice is impossible because the eigenvectors are not known exactly. To simulate this uncertainty, let us choose a force vector that is more parallel to the last two eigenpairs:

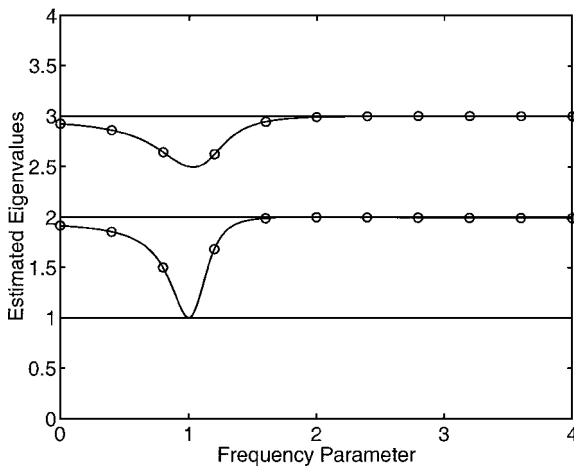
$$\mathbf{f} = \mathbf{M} (0.1 \phi_1 + 0.45 \phi_2 + 0.45 \phi_3) \quad (55)$$

Because this force is dominated by the last two eigenvectors, we expect that the condensation will yield estimates of these eigenpairs. Let us again fix one of the frequency parameters at  $\lambda = 2.5$  and vary the second. The resulting eigenvalue estimates are shown in Fig. 3. The force vector is chosen to favor the  $n = 2$  and 3 eigenpairs. In contrast to Fig. 2, good estimates of the first two eigenvalues are obtained as long as the frequency is greater than 1.5.

Figures 1 and 2 can be viewed as worst cases because the force vector is equally parallel to all eigenvectors, whereas Fig. 3 is near the best case because the force vector is nearly orthogonal to the third eigenvector. It is interesting to study cases between these two extremes by allowing elements of the force vector to be random variables. In particular, let the force be a vector of normally distributed random numbers with a zero mean and unity standard deviation. The resulting eigenvalue estimates are shown in Fig. 4. The general trend



**Fig. 2** Eigenvalues estimated by a condensation with one frequency parameter at  $\lambda = 2.5$  and the other parameter varied along the abscissa; horizontal lines represent exact eigenvalues.



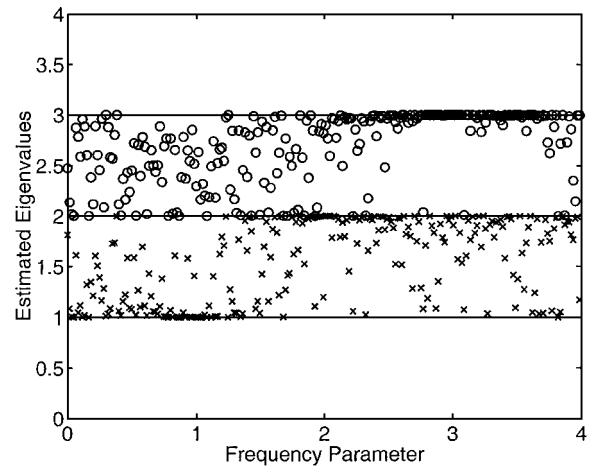
**Fig. 3** Eigenvalues estimated by a condensation with one frequency parameter at  $\lambda = 2.5$  and the other parameter varied along the abscissa; horizontal lines represent exact eigenvalues.

is that an eigenvalue's estimate is accurate when the frequency parameter is near it; however, no eigenvalue is consistently estimated.

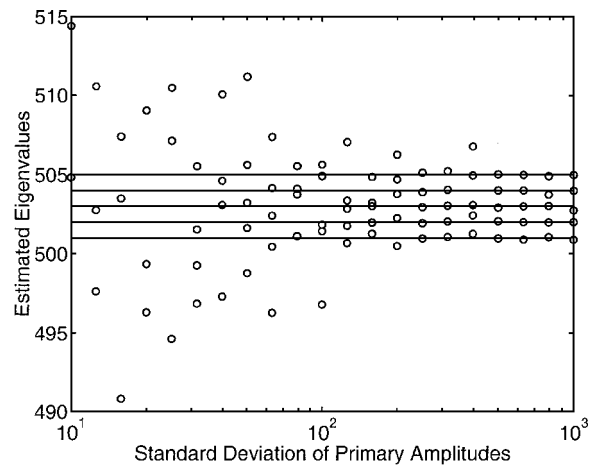
From Figs. 1–3, we may conclude that a single condensation works best when an approximate knowledge of the eigenvectors is used to construct the force vectors. When such knowledge exists, Fig. 3 indicates that the condensation accurately estimates the eigenvalue as long as the frequency parameter is in the neighborhood of an eigenvalue. This result is explained in Eq. (26), which indicates the retention of the primary eigenpairs when the primary and secondary amplitudes satisfy Eqs. (24) and (25).

Now let us examine these condensation properties for a structure with 1000 degrees of freedom and evenly spaced eigenvalues defined by  $\lambda_n = n$ . The eigenvector elements are normally distributed random variables with zero mean and unity standard deviation. Suppose that we seek estimates of the five eigenvalues near the middle of the range,  $n = 501$ –505, by performing a condensation with five forced response vectors. However, in this example we shall combine the effects of choosing the frequency parameter and force vector by simply specifying the amplitudes of the eigenpairs for each of the forced response vectors. The five targeted eigenpairs are the primary ones, and the remaining 995 eigenpairs are the secondary ones.

We assume that some approximate knowledge of the primary eigenpairs exists that allows the primary amplitudes to exceed the secondary amplitudes. The secondary amplitudes are taken as normally distributed random variables with zero mean and unity standard deviation. The primary amplitudes are also normally distributed random variables with zero mean; however, the standard deviation is varied to simulate uncertainties in the primary eigenpairs. Each



**Fig. 4** Eigenvalues estimated by a condensation with one frequency parameter at  $\lambda = 2.25$  and the other parameter varied along the abscissa; horizontal lines represent exact eigenvalues.



**Fig. 5** Eigenvalue estimates obtained by using five response vectors; horizontal lines placed at the exact eigenvalues.

time a standard deviation of the primary amplitudes is chosen, one numerical experiment is performed in which the matrix  $\mathbf{C}$ , which holds the primary and secondary amplitudes, is randomly generated. This matrix is used to condense the mass and stiffness matrices, according to Eqs. (19) and (20), and the smaller eigenvalue problem is solved. The results are shown in Fig. 5. When the standard deviation exceeds 100, the condensation accurately retains most of the eigenpairs. All of the eigenpairs are accurately estimated when the standard deviation exceeds 1000.

These results were obtained by choosing the number of response vectors to match the number of targeted eigenpairs, which was five. However, one might expect the results to improve if more response vectors are used. To investigate this, the preceding example was analyzed using 25 forced response vectors instead of 5. The results of this analysis are shown in Fig. 6, where one observes that the condensation accurately retains all of the eigenpairs when the standard deviation exceeds 100. The standard deviations of the secondary eigenpairs are unity. The additional response vectors have substantially improved the accuracy of the condensation. The additional response vectors also produced more incorrect eigenvalue estimates, but those could be identified by evaluating the error bounds in Eq. (51).

When the eigenpairs are not known well enough to construct condensations that retain them, the iterative MRQI procedure that was already described may be used to find all of the eigenvalues in a specified range. To demonstrate the procedure, let us consider a random structure whose eigenvalues fall randomly but with uniform distribution in the following three ranges:  $0 \leq \lambda \leq 1$ ,  $1 < \lambda \leq 2$ , and  $2 < \lambda \leq 3$ . Furthermore, we assume that there are 10 eigenvalues in

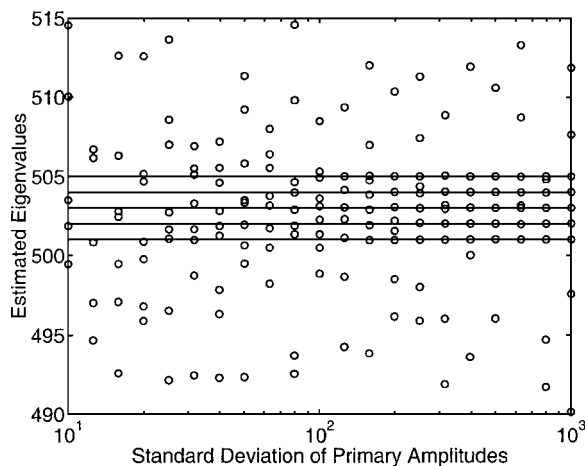


Fig. 6 Eigenvalue estimates obtained by using 25 response vectors; horizontal lines placed at exact eigenvalues.

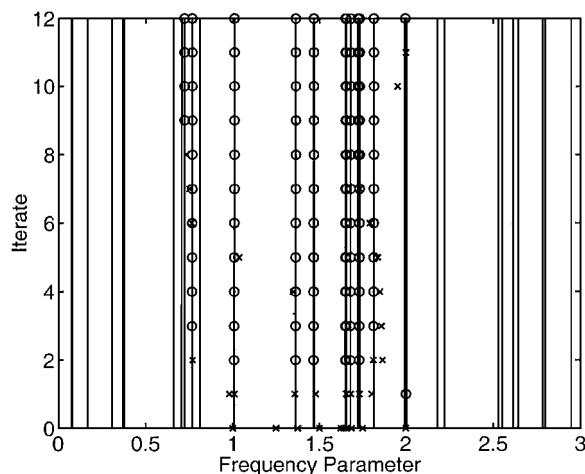


Fig. 7 Eigenvalue estimates obtained over the range  $1 \leq \lambda < 2$  by the iterative procedure; vertical lines placed at exact eigenvalues and O and x placed at the estimated eigenvalues, with O denoting converged values.

each range and that the elements of the eigenvectors are normally distributed as before. Let us seek to find all of the eigenvalues in the middle range,  $1 < \lambda \leq 2$ .

The results of this procedure are shown in Fig. 7, where several interesting features are observed. Before the iteration begins, the frequency parameter is sampled according to the bisection method, so that the sampled values of the frequency parameter lie as close as possible to the eigenvalues. For example, notice that frequency parameters are clustered near the group of eigenvalues around  $\lambda = 1.7$ . However, the frequency parameters are more evenly distributed below  $\lambda = 1.5$  because the eigenvalues are more evenly distributed in this range. As the iteration proceeds, the condensation finds eigenvalues that lie outside the specified band. For example, the procedure finds the eigenvalue near  $\lambda = 0.75$ . As it finds these eigenpairs, the procedure effectively removes them from the search space and begins to converge on eigenvalues that lie in the range. Initially the method converges very quickly, with roughly half the estimates converging within each iteration. After the second iteration, for example, 9 of our 10 original eigenvalue estimates have converged to the correct eigenvalues. One of these converged values, however, lies outside the frequency range of interest. Finding the last eigenvalue in the range of interest takes a relatively large number of iterations. Eventually, the procedure converges to all of the eigenvalues in the range.

Our experience with other simulations has indicated that the eigenvalues near the edges of the frequency range usually converge last. Usually, this is because the algorithm converges to a nearby eigenvalue that is outside the range, at the expense of finding one inside the range. This slowness may be improved by initially con-

sidering a frequency range that is somewhat larger than the range in which one would like to find all eigenvalues.

## Conclusions

A condensation has been proposed that uses forced response vectors to condense the generalized eigenvalue problem found in structural vibrations. We have shown applications of the condensation both as a postprocessing step to aid interpretation of structural dynamics predictions and also as the kernel of an iterative scheme to compute eigenpairs. We have demonstrated that the success of the condensation in extracting a subset of eigenpairs depends critically on the choices of force vectors and frequency parameters. In cases where a subset of eigenpairs is approximately known, the condensation can accurately estimate them in one step. If the eigenpairs are not approximately known, an iterative procedure (MRQI) has been developed that finds all of the eigenvalues in a specified range.

We developed analytical expressions for error bounds on the estimated eigenvalues that require very little computational effort. These are useful for both applications of the condensation considered. Their application to the iterative procedure is obvious. In the single-step postprocessing application, it is useful to know whether the eigenvalue predictions are actually accurate or not, lest too much faith is invested in their precise values.

One question that has not been addressed here is that of the efficiency of the proposed condensation because forced response vectors are computationally expensive for large systems. Although a detailed efficiency study has not been conducted, it is clear that the condensation is extremely economical in cases where the forced response vectors have been computed for other purposes. Regarding the efficiency of the iterative convergence, we have demonstrated analytically the known cubic convergence rate associated with RQI, which MRQI naturally inherits.

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